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Extreme points of the unit ball of the algebra generated by composition operators

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Abstract

We study the extreme points of the unit ball of the algebra generated by composition operators on the disk algebra.

1 Introduction

Let $\mathbb{D}$ be the open unit disk. We denote by $\overline{\mathbb{D}}$ its closure and by $\partial\mathbb{D}$ its boundary. Let $H(\mathbb{D})$ be the set of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ be the set of all analytic self-map of $\mathbb{D}$. Every analytic self-map $\varphi \in S(\mathbb{D})$ the composition operator C_φ on $H(\mathbb{D})$ defined by

$$C_\varphi f(z) = f(\varphi(z)).$$

Let H^∞ be the set of all bounded analytic functions on $\mathbb{D}$. Then H^∞ is a Banach algebra with the supremum norm,

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

Every composition operator is bounded on H^∞ and $\|C_\varphi\| = 1$. It is known that C_φ is compact on H^∞ if and only if $\|\varphi\|_\infty < 1$.

Recall that the disk algebra A is the Banach algebra of all functions analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ with the supremum norm. To define C_φ on A, we need the condition $C_\varphi z = \varphi \in A$. Denote by $S(\overline{\mathbb{D}})$ the closed unit ball of A. Then every $\varphi \in S(\overline{\mathbb{D}})$ induces C_φ which acts on A. If φ is a constant function with value $\omega \in \partial\mathbb{D}$, then φ is not in $S(\mathbb{D})$ but in $S(\overline{\mathbb{D}})$. We denote that $\mathbb{T} = \{\varphi \equiv \omega \in \partial\mathbb{D}\}$. By the maximum modulus principle, it is shown that $S(\overline{\mathbb{D}}) \setminus \mathbb{T} = S(\mathbb{D}) \cap A$. Similarly to the case of H^∞, we can see that $\|C_\varphi\|_A = 1$ for every $\varphi \in S(\overline{\mathbb{D}})$ and C_φ is compact on A if and only if $\|\varphi\|_\infty < 1$ or $\varphi \equiv e^{i\theta}$.

Let $\mathcal{X}$ be an analytic functional Banach space on $\mathbb{D}$, that is, each element is analytic on $\mathbb{D}$ and the evaluation at each point of $\mathbb{D}$ is a non-zero bounded linear functional on $\mathcal{X}$. Let $\mathcal{C}(\mathcal{X})$ be the collection of all bounded composition operators on $\mathcal{X}$, endowed with the operator norm topology. Originally this topic was posed for the case of $\mathcal{C}(H^2)$ by Shapiro and Sundberg in [7]. They raised the following three problems: (i) Characterize the path components of $\mathcal{C}(H^2)$. (ii) Which composition operators are isolated in $\mathcal{C}(H^2)$? (iii) Which differences of composition operators are compact on H^2? These problems are still open. In [6], MacCluer, Ohno and Zhao solved (i) and (ii) of the problems above for $\mathcal{C}(H^\infty)$.

Their results was descrived by the terms of the pseudo-hyperbolic distance on $\mathbb{D}$. For $p \in \mathbb{D}$, let α_p be the automorphism of $\mathbb{D}$ exchanging 0 for p. Then α_p has the following form;

$$\alpha_p(z) = \frac{p - z}{1 - \overline{p}z}.$$

The pseudo-hyperbolic distance $\rho(z, w)$ between z and w in $\mathbb{D}$ is defined by

$$\rho(z, w) = |\alpha_z(w)| = \left| \frac{z - w}{1 - \overline{z}w} \right|.$$

Here we define the induced distance d_ρ on $S(\mathbb{D})$, that is,

$$d_\rho(\varphi, \psi) = \sup_{z \in \mathbb{D}} \rho(\varphi(z), \psi(z))$$

for φ and ψ in $S(\mathbb{D})$. In [6] the operator norms of the differences of composition operators on H^∞ are estimated as following;

$$\|C_\varphi - C_\psi\| = \frac{2 - 2\sqrt{1 - d_\rho(\varphi, \psi)^2}}{d_\rho(\varphi, \psi)}. \tag{1}$$

Hence $\mathcal{C}(H^\infty)$ can be identified with the space $S(\mathbb{D}, d_\rho)$. We denote $C_\varphi \sim_{\mathcal{X}} C_\psi$ if they are in the same component of $\mathcal{C}(\mathcal{X})$. In [6], it is proved that $C_\varphi \sim_{H^\infty} C_\psi$ if and only if $d_\rho(\varphi, \psi) < 1$.

Let $\mathcal{Y}$ be a convex subset of a locally convex space. We recall that an element y of $\mathcal{Y}$ is called an extreme point of $\mathcal{Y}$ if the conditions $0 < r < 1$, $y_1, y_2 \in \mathcal{Y}$ and $y = (1-r)y_1 + ry_2$, implies that $y_1 = y_2 = y$. For a normed space $\mathcal{Z}$, we denote by $U_{\mathcal{Z}}$ the cloed unit ball of $\mathcal{Z}$. By Rudin-de Leeuw's Theorem([4, Ch.9]), φ is an extreme point of U_{H^∞} if and only if

$$\int_0^{2\pi} \log(1 - |\varphi(e^{i\theta})|) d\theta = -\infty. \tag{2}$$

MacCluer, Ohno and Zhao proved that if C_φ is isolated in $\mathcal{C}(H^\infty)$, then φ is an extreme point of U_{H^∞}. In [5], the converse was proved. We remark that the connected components of $\mathcal{C}(H^\infty)$ are characterized by a equivalence relation which is in the similar form of the Gleason parts of the maximal ideal space of H^∞. In this sense, the isolated points of $\mathcal{C}(H^\infty)$ corresponds to the single Gleason parts.

The topological structure of $\mathcal{C}(A)$ is similar to that of $\mathcal{C}(H^\infty)$. To introduce such results, we extend the pseudo-hyperbolic distance to $\overline{\mathbb{D}}$ as following; For $z \in \partial\mathbb{D}$ and $w \in \overline{\mathbb{D}}$ such that $z \neq w$, define that $\rho(z,z) = 0$ and $\rho(z,w) = 1$. Hence the induced distance d_ρ is defined on $S(\overline{\mathbb{D}})$. We remark that φ is extreme point of the closed unit ball $S(\overline{\mathbb{D}})$ of A if and only if the condition (2) holds (see [4, p. 139]). We denote that $\mathcal{K} = \{C_\varphi \text{ is compact on } A\}$ and $\Delta = \{C_\varphi \in \mathcal{C}(A) : \varphi \equiv \omega \in \partial\mathbb{D}\}$. Now the results on the topological structure of $\mathcal{C}(H^\infty)$ can be applied on $\mathcal{C}(A)$ by the similar proof in [5] and [6].

Theorem 1.1 *Let C_φ, C_ψ be in $\mathcal{C}(A)$. Then*

(i) $\|C_\varphi - C_\psi\|_A = \dfrac{2 - 2\sqrt{1 - d_\rho(\varphi,\psi)^2}}{d_\rho(\varphi,\psi)}.$

(ii) $C_\varphi \sim_A C_\psi$ if and only if $\|C_\varphi - C_\psi\|_A < 2$.

(iii) The following are equivalent:

(a) C_φ is isolated in $\mathcal{C}(A)$.

(b) For all $C_\psi \neq C_\varphi$, $\|C_\varphi - C_\psi\|_A = 2$.

(c) φ is an extreme point of the closed unit ball of A.

(d) $\displaystyle\int_0^{2\pi} \log(1 - |\varphi(e^{i\theta})|)d\theta = -\infty.$

(iv) Every $C_\varphi \in \Delta$ is compact on A and isolated in $\mathcal{C}(A)$.

(v) $\mathcal{K} \setminus \Delta$ is a component of $\mathcal{C}(A)$.

Denote by $\mathrm{Comp}_{\mathcal{X}}(\varphi)$ the path component of $\mathcal{C}(\mathcal{X})$ which contains C_φ. Then we can immediately get the following corollary, which mentions the relation between the topological structure of $\mathcal{C}(A)$ and that of $\mathcal{C}(H^\infty)$.

Corollary 1.2 *Let C_φ and C_ψ be in $\mathcal{C}(A) \setminus \Delta$. Then we have the following.*

(i) $\mathrm{Comp}_A(\varphi) = \mathrm{Comp}_{H^\infty}(\varphi) \cap \mathcal{C}(A)$.

(ii) $C_\varphi \sim C_\psi$ *in* $\mathcal{C}(A)$ *if and only if* $C_\varphi \sim C_\psi$ *in* $\mathcal{C}(H^\infty)$.

(iii) C_φ *is isolated in* $\mathcal{C}(A)$ *if and only if* C_φ *is isolated in* $\mathcal{C}(H^\infty)$.

In general, $\mathcal{C}(\mathcal{X})$ is a semigroup with respect to the products, but the finite linear combinations of composition operators are not in $\mathcal{C}(\mathcal{X})$. We denote by $\langle \mathcal{C}(\mathcal{X}) \rangle$ the collection of all finite linear combinations of composition operators on $\mathcal{X}$. Let $\mathcal{L}(\mathcal{X})$ denote the operator norm closure of $\langle \mathcal{C}(\mathcal{X}) \rangle$. In the next section, we investigate the relation between the isolated points of $\mathcal{C}(A)$ and the extreme points of $U_{\mathcal{L}(A)}$. Our main result states that C_φ is a extreme point of $\mathcal{L}(A)$ if and only if C_φ is a isolated point of $\mathcal{C}(A)$.

2 Extreme point of $U_{\mathcal{L}(A)}$

At first, we observe that composition operators are linearly independent each other in $\langle \mathcal{C}(A) \rangle$.

Proposition 2.1 *Let $\varphi_1, \cdots, \varphi_n$ be the distinct analytic maps of $S(\overline{\mathbb{D}})$ and let $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$. If $\lambda_1 C_{\varphi_1} + \cdots + \lambda_n C_{\varphi_n}$ is the zero operator on A, then $\lambda_1 = \cdots = \lambda_n = 0$.*

In [3], Gorkin and Mortini investigated the norms and essential norms of finite linear combinations of composition operators. They also proved that $\langle \mathcal{C}(A) \rangle$ is not closed. and the multiplication operator M_z is not contained in $\mathcal{L}(A)$. Here we will construct an example of elements of $\mathcal{L}(A) \setminus \langle \mathcal{C}(A) \rangle$. For a continuous curve $\{C_{\varphi_t}\}_{t\in[0,1]}$ in $\mathcal{C}(A)$, we define that

$$T_n = \sum_{k=1}^{n} \frac{1}{n} C_{\varphi_{\frac{k}{n}}}.$$

Then $\|T_n\| = 1$. For $f \in A$ and $p \in \mathbb{D}$, we have that

$$T_n f(p) = \sum_{k=1}^{n} \frac{1}{n} f(\varphi_{\frac{k}{n}}(p)) \to \int_0^1 f(\varphi_t(p))dt$$

as $n \to \infty$. Since $\{T_n f\}$ is Cauchy sequence in A, we have that

$$\int_0^1 f(\varphi_t(z))dt \in H^\infty.$$

Here we denote by I_{φ_t} the following integral operator:

$$I_{\varphi_t} f(z) = \int_0^1 f(\varphi_t(z))dt. \tag{3}$$

Then the Banach-Steinhaus Theorem implies the following lemma.

Lemma 2.2 *If $\{C_{\varphi_t}\}_{t\in[0,1]}$ is a continuous curve in $\mathcal{C}(A)$, then the corresponding integral operator I_{φ_t} is in $U_{\mathcal{L}(A)}$.*

Example 2.3 *(i) Suppose that $C_\varphi \sim_A C_\psi$. Put $\varphi_t = (1-t)\varphi + t\psi$. Then $\{C_{\varphi_t}\}_{t\in[0,1]}$ is a continuous curve in $\mathcal{C}(H^\infty)$ (see [6]) and*

$$I_{\varphi_t} f(z) = \frac{F(\psi(z)) - F(\varphi(z))}{\psi(z) - \varphi(z)}$$

where $F(z)$ is the primitive function of $f(z)$.

(ii) Suppose that $\|\varphi\|_\infty < 1$. Choose a positive number r such that $r < 1 - \|\varphi\|_\infty$. We define that $\varphi_t(z) = \varphi(z) + re^{2\pi it} z$. Then $\|\varphi_t\|_\infty < 1$ for all t. Since every $\varphi_t(\mathbb{D})$ is a compact subset of $\mathbb{D}$, $d_\rho(\varphi_s, \varphi_t) \to 0$ as $s \to t$. Thus $\{C_{\varphi_t}\}_{t\in[0,1]}$ is a closed continuous curve in $\mathcal{C}(H^\infty)$. By the Cauchy's Formula, we have that $I_{\varphi_t} = C_\varphi$.

We remark that the condition $\|\varphi\|_\infty < 1$ induces that C_φ is not an extreme point of $U_{\mathcal{L}(A)}$. From (ii) of Example 2.3, we have that, for $f \in A$ and $p \in \mathbb{D}$,

$$C_\varphi f(p) = \int_0^{\frac{1}{2}} f\left(\varphi(p) + rp\, e^{2\pi it}\right) dt + \int_{\frac{1}{2}}^1 f\left(\varphi(p) + rp\, e^{2\pi it}\right) dt$$

Let $\sigma_t(z) = \varphi(z) + re^{\pi it} z$ and $\tau_t(z) = \varphi(z) - re^{\pi it} z$. By changing variables,

$$C_\varphi = \frac{1}{2} I_{\sigma_t} + \frac{1}{2} I_{\tau_t}. \tag{4}$$

Since $I_{\sigma_t} \neq I_{\tau_t}$, we can conclude that C_φ is not an extreme point. Then we have the following.

Proposition 2.4 *If C_φ is compact on A, then C_φ is not an extreme point of $U_{\mathcal{L}(A)}$.*

Here we state our main result.

Theorem 2.5 *C_φ is an extreme point of $U_{\mathcal{L}(A)}$ if and only if C_φ is an isolated point of $\mathcal{C}(A)$.*

We remark that the same proof of the "only if" part can be applied to $\mathcal{L}(H^\infty)$. We here present two problems.

Problem *(i) Can Theorem 2.5 be applied to $\mathcal{L}(H^\infty)$?*

(ii) Is there other extreme point of the closed unit ball of $\mathcal{L}(A)$?

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